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# TECHNICAL NOTES

## Steady-state moisture profiles in an unsaturated porous medium with impermeable boundaries

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### INTRODUCTION

MOISTURE migration in unsaturated porous media has been extensively studied by many investigators [1–6]. Many insulation and structural materials with moisture shields have impermeable boundaries. The objective of this work is to investigate a methodology to predict the steady-state moisture profile using the steady-state temperature profile for unsaturated porous media with impermeable boundaries. In the present analysis the porous medium is assumed to be homogeneous, isotropic, and the thermophysical properties of the medium are assumed to remain constant. The heat transfer in the medium is assumed to be solely due to molecular conduction and the effect of gravity on the moisture migration is neglected. The heat transfer mechanism due to phase change of the moisture such as freezing and boiling are neglected. The total pressure in the porous medium is assumed to be a constant.

### ANALYSIS

The temperature ($T$) and moisture concentration ($W$) fields in an arbitrary three-dimensional unsaturated porous medium satisfy the following diffusion equations:

$$\frac{\partial T}{\partial \tau} = \alpha \nabla^2 T \tag{1}$$

$$\frac{\partial W}{\partial \tau} = D_T \nabla^2 T + D_M \nabla^2 W. \tag{2}$$

Moisture concentration flux ($J$) is given by $J = -D_T \nabla T - D_M \nabla W$. The above equations (1) and (2) can be written in dimensionless forms as

$$\frac{\partial \Psi_T}{\partial Fo} = \bar{\nabla}^2 \Psi_T \tag{3}$$

$$\frac{1}{Lu}\frac{\partial \Psi_M}{\partial Fo} = \bar{\nabla}^2 \Psi_T + \bar{\nabla}^2 \Psi_M,$$

$$\text{with } \frac{\partial \Psi_M}{\partial n} = -\frac{\partial \Psi_T}{\partial n} \text{ on the boundary} \tag{4}$$

where

$$\bar{\nabla}^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2 + \partial^2/\partial Z^2.$$

The conservation of mass requires that

$$\int \Psi_M \, dV = 0 \tag{5}$$

due to the nature of the impermeable boundary. The boundary condition for moisture equation (4) is not homogeneous. To simplify the solution process, the boundary condition is made homogeneous by introducing the following transformation:

$$\Psi = \Psi_M + \Psi_T. \tag{6}$$

Note that function $\Psi$ has no direct physical meaning but is an intermediate mathematical entity to obtain the moisture profile. Accordingly, the moisture equation takes the form

$$\frac{1}{Lu}\frac{\partial \Psi}{\partial Fo} - \bar{\nabla}^2 \Psi = \frac{1}{Lu}\frac{\partial \Psi_T}{\partial Fo} \tag{4a}$$

with $\partial \Psi / \partial n = 0$ on the boundary.

### STEADY-STATE PROFILES

If steady-state moisture and temperature profiles exist for a prescribed thermal boundary condition, equations (3) and (4) under steady-state conditions reduce to

$$\bar{\nabla}^2 \Psi_T = 0 \quad \text{and} \quad \bar{\nabla}^2 \Psi_M = 0. \tag{7}$$

Invoking Green's theorem

$$\int (\Psi_T \bar{\nabla}^2 \Psi_M - \Psi_M \bar{\nabla}^2 \Psi_T) \, dV = \oint \left\{ \Psi_T \left( \frac{\partial \Psi_M}{\partial n} \right) - \Psi_M \left( \frac{\partial \Psi_T}{\partial n} \right) \right\} dS. \tag{8}$$

Substituting equation (7) into equation (8) one obtains

$$\oint \Psi_T \left( \frac{\partial \Psi_M}{\partial n} \right) dS = \oint \Psi_M \left( \frac{\partial \Psi_T}{\partial n} \right) dS.$$

Combining the impermeable boundary condition with the above equation, we arrive at

$$\oint \Psi_T \left( \frac{\partial \Psi_T}{\partial n} \right) dS = \oint \Psi_M \left( \frac{\partial \Psi_M}{\partial n} \right) dS. \tag{9}$$

## NOMENCLATURE

| | |
|---|---|
| $a_x, a_y, a$ | dimensionless lengths in $x$-, $y$- and $z$-directions ($a_x = 1$, $a_y = L_y/L_x$ and $a_z = L_z/L_x$) |
| $A$ | constant $\langle\Psi_T\rangle$ |
| $D_T$ | thermal mass diffusion coefficient [$m^2 s^{-1} °C^{-1}$] |
| $D_M$ | moisture mass diffusion coefficient [$m^2 s^{-1}$] |
| $Fo$ | Fourier number, $\alpha\tau/L_x^2$ |
| $J$ | moisture concentration flux |
| $K$ | thermal conductivity [$W m^{-1} °C^{-1}$] |
| $L_x, L_y, L_z$ | lengths in $x$-, $y$- and $z$-directions |
| $Lu$ | Luikov number, $D_M/\alpha$ |
| $n$ | unit outward normal to the surface |
| $q_0$ | heat flux [$W m^{-2}$] |
| $\bar{q}_0$ | dimensionless heat flux, $L_x q_0/KT_i$ |
| $S$ | dimensionless surface area of the porous medium |
| $T$ | temperature field |
| $\Delta T$ | reference temperature difference |
| $T_i$ | initial temperature of the porous medium |
| $v$ | volume of the porous medium, $L_x L_y L_z$ |
| $V$ | dimensionless volume of the porous medium, $a_x a_y a_z$ |
| $W$ | moisture concentration, $\rho_l/\rho_s$ |
| $W_i$ | initial moisture concentration in the medium |
| $x, y, z$ | space variables |
| $X, Y, Z$ | dimensionless space variables ($X = x/L_x$, $Y = y/L_y$ and $Z = z/L_z$). |

Greek symbols

| | |
|---|---|
| $\alpha$ | thermal diffusivity [$m^2 s^{-1}$] |
| $\rho_l$ | liquid density [$kg m^{-3}$] |
| $\rho_s$ | dry medium density [$kg m^{-3}$] |
| $\tau$ | time variable |
| $\Psi$ | $\Psi_T + \Psi_M$ |
| $\Psi_M$ | dimensionless moisture concentration, $D_M(W - W_i)/D_T\Delta T$ |
| $\Psi_T$ | dimensionless temperature, $(T - T_i)/\Delta T$ |
| $\Psi_\infty$ | asymptote of $\Psi$. |

Other symbols

| | |
|---|---|
| $\langle\ \rangle$ | volume average |
| $\nabla^2$ | Laplacian operator |
| $\bar{\nabla}^2$ | Laplacian operator in dimensionless coordinates, equation (4). |

Since equation (9) is valid for any arbitrary geometry and thermal boundary condition, we finally obtain that $\Psi_M = -\Psi_T + A$, where $A$ is a constant. This can be verified by back substitution. Using the equation for mass conservation (5), the constant $A$ is identified as

$$A = \frac{1}{V}\int \Psi_T \, dV = \langle\Psi_T\rangle$$

where $\langle\Psi_T\rangle$ represents the volume average of $\Psi_T$. From the aforementioned discussion we arrive at the following remark.

*The non-dimensional steady-state moisture* ($\Psi_M$) *and temperature* ($\Psi_T$) *profiles are related by* $\Psi_M = -\Psi_T + \langle\Psi_T\rangle$, *irrespective of the thermal boundary conditions, domain geometry, and dimensionality of an unsaturated porous medium with impermeable boundaries, provided that the steady-state temperature profile* ($\Psi_T$) *exists.*

This suggests that, if the ultimate or steady-state moisture profile is of main concern, one needs only to find the steady-state temperature profile, which is a solution to the Laplace equation (7).

## DRYOUT IN POROUS MEDIA

We turn our attention now to the concept of dryout in the porous medium, when the moisture profile has reached its steady state. The 'dryout' region in the porous medium is defined as the region where the moisture concentration ($W$) is zero.

Combining the outcome of our analysis and the original definition, we obtain

$$(\Psi_M)_{s.s} = \frac{D_M}{D_T\Delta T}((W)_{s.s} - W_i) = -(\Psi_T)_{s.s} + \langle(\Psi_T)_{s.s}\rangle$$

or

$$(W)_{s.s} = W_i - \frac{D_T\Delta T}{D_M}((\Psi_T)_{s.s} - \langle(\Psi_T)_{s.s}\rangle). \quad (10a)$$

From this, we define the wet region ($W > 0$) as

$$[\Psi_T(X, Y, Z)]_{s.s} < \frac{D_M W_i}{D_T\Delta T} + \langle(\Psi_T)_{s.s}\rangle. \quad (10b)$$

If the region defined by equation (10b) occupies the whole domain, then there does not exist a dried region. However, if this region occupies a partial domain, then an ultimate dryout exists in the medium.

## APPLICATIONS

Consider a one-dimensional porous medium of $L_x = 0.5$ m at an initial temperature of $T_i$ and an initial moisture concentration of $W_i$.

*End walls at uniform temperatures*

In the first application of the analysis, consider the case where the left-hand wall is heated to 70°C, while the right-hand wall is maintained at the initial temperature, $T_i = 20°C$. The temperature field is obtained from

$$\frac{\partial\Psi_T}{\partial Fo} = \frac{\partial^2\Psi_T}{\partial X^2} \quad \left| \begin{array}{l}\text{Initial and boundary conditions:} \\ \Psi_T = 0, \ Fo = 0 \\ \Psi_T = 0, \ X = 1 \\ \Psi_T = 1, \ X = 0 \ (\Delta T = 70-20)\end{array}\right. .$$

Using the Laplace transform technique, the solution is found to be

$$\Psi_T = 1 - X - 2\sum_1^\infty \frac{\sin m\pi x}{m\pi}\exp(-m^2\pi^2 Fo).$$

The steady-state temperature profile becomes

$$(\Psi_T)_{s.s} = 1 - X, \quad \text{with} \quad \langle(\Psi_T)_{s.s}\rangle = 1/2.$$

The present analysis provides the steady-state moisture profile as

$$(\Psi_M)_{s.s} = -(1-X) + \langle 1-X\rangle = -(1-X) + 1/2 = X - 1/2.$$

This also satisfies the mass conservation equation (5). Reverting to the dimensional variable $x$, ($X = 2x$), equation (10a) yields

$$W = W_i + (2x - 1/2)\frac{D_T\Delta T}{D_M}. \quad (11)$$

The wet region is given by equation (10b) as

$$\frac{1}{4} - \frac{D_M W_i}{2D_T\Delta T} < x.$$

The possibility of the existence of a dryout region depends on the initial concentration, $W_i$. The extent of the dryout region increases with a decrease in $W_i$. When the left-hand side of the above inequality is set equal to zero, we obtain

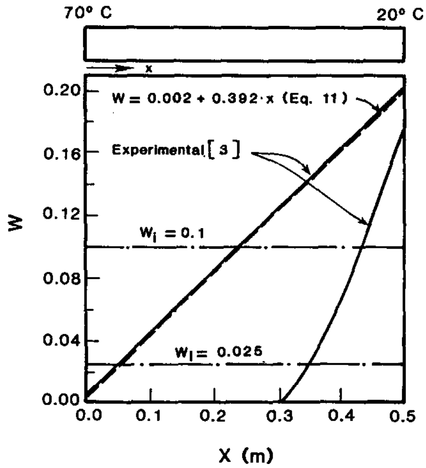


FIG. 1. Steady-state moisture profile obtained by Krischer and Rohnalter (Fig. 8 of ref. [3]).

the following upper bound of $W_i$ which can result in an ultimate dryout. (Dryout would start from the left-hand wall in this example case)

$$(W_i)_{\text{dryout}} = \frac{D_T \Delta T}{2D_M}. \qquad (12)$$

Note that if $W_i > (W_i)_{\text{dryout}}$, no dryout is possible.

The theoretical results obtained in this investigation are compared with the experimental data obtained by Krischer and Rohnalter (Fig. 8 of ref. [3]) for the same thermal and mass boundary conditions. They investigated the steady-state moisture profiles for initial concentrations of $W_i = 10$ and 2.5%. The present results compare very well and are shown in Fig. 1.

*End walls at uniform heat flux and uniform temperature*

In this case, the left-hand wall is subjected to a constant heat flux, $q_0$, and the right-hand wall is kept at a constant temperature, $T_i$. The governing equation for the temperature field and the associated boundary conditions are

$$\frac{\partial \Psi_T}{\partial Fo} = \frac{\partial^2 \Psi_T}{\partial X^2} \quad \begin{array}{|l} \text{Initial and boundary conditions:} \\ \Psi_T = 0, \quad Fo = 0 \\ \Psi_T = 1, \quad X = 1 \\ \partial\Psi_T/\partial X = -\bar{q}_0, X = 0 \; (\Delta T = T_i) \end{array}.$$

Using the Laplace transform technique, the solution is found to be

$$\Psi_T = \bar{q}_0(1-X) + 2\bar{q}_0 \sum_{n=1}^{\infty} \frac{(-1)^n \sin[(n-1/2)\pi(1-X)]}{(n-1/2)^2\pi^2} \times \exp[-(n-1/2)^2\pi^2 Fo].$$

The steady-state temperature profile is

$$(\Psi_T)_{s.s} = \bar{q}_0(1-X).$$

The steady-state moisture profile then is given by

$$(\Psi_M)_{s.s} = -\bar{q}_0(1-X) + \langle \bar{q}_0(1-X) \rangle = \bar{q}_0(X-1/2).$$

This moisture profile also readily satisfies the mass conservation equation (5).

Reverting to the dimensional variable ($X = 2x$), we obtain from equation (10a)

$$W = W_i + (2x - 1/2)\frac{D_T L_x q_0}{D_M K}. \qquad (13)$$

The wet region is given by equation (10b) as

$$\frac{1}{4} - \frac{D_M W_i K}{2D_T L_x q_0} < x.$$

Following the same reasoning as before, the upper bound of $W_i$ is obtained as

$$(W_i)_{\text{dryout}} = \frac{D_T L_x q_0}{2D_M K} \qquad (14)$$

so that if $(W_i) > (W_i)_{\text{dryout}}$, no dryout is possible.

## SUMMARY

The steady-state temperature profile discussed hitherto, refers only to a truly steady-state condition, independent of the time variable. Mathematically, this is equivalent to saying that the poles of a Laplace transformed profile should lie wholly at the origin and the left-half of the imaginary axis.

A mathematical result has been derived that is related to the heat and mass transfer in a homogeneous, isotropic, and unsaturated porous medium with impermeable boundaries. In the present analysis, the thermophysical properties of the medium are assumed to remain constant and the effect of gravity is neglected. This analysis establishes a method of predicting the existence of an ultimate dryout region and the steady-state moisture profile, knowing only the steady-state temperature profile. Equations (12) and (14) can be used to determine the possibility of the existence of a dryout region for given material properties, initial moisture concentration and thermal boundary conditions. The result obtained in this analysis is not applicable to the case of a thermal boundary condition which does not yield a steady-state temperature profile.

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